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Proof: The center of perspective, P_3 , of the two given circles, O_1 and O_2 , has the same power with respect to all circles tangent to these two in a given way. Therefore, the locus of the point of contact of any two such circles which are tangent to each other is a circle orthogonal to them all.

THEOREM: The circle with P_3 as center and P_3A as radius cuts every C_n orthogonally.

Proof: Let there be drawn with T as center and TA as radius a circle. This will pass through C_1 and C_2 . Therefore $C_1C_2P_3$ will be the radical axis of the circle just drawn and C_n . The circle with P_3 as center and P_3A as radius is orthogonal to the circle with center T . It is therefore orthogonal to every C_n .

SOME VANISHING AGGREGATES CONNECTED WITH CIRCULANTS.

By W. H. METZLER, Syracuse University.

In the course of certain investigations on Lagrange's Equation for circulants¹ by Dr. Muir² in 1912 attention was called to the vanishing aggregate:

$$\begin{vmatrix} 1 & b & c & d \\ 1 & c & d & e \\ 1 & d & e & a \\ 1 & e & a & b \end{vmatrix} + \begin{vmatrix} 1 & b & c & e \\ 1 & c & d & a \\ 1 & d & e & b \\ 1 & a & b & d \end{vmatrix} + \begin{vmatrix} 1 & b & d & e \\ 1 & c & e & a \\ 1 & e & b & c \\ 1 & a & c & d \end{vmatrix} + \begin{vmatrix} 1 & c & d & e \\ 1 & e & a & b \\ 1 & a & b & c \\ 1 & b & c & d \end{vmatrix} = 0,$$

where a, b, c, d, e are the elements of a circulant of order five. He obtained it as the coefficient of the first power of x in Lagrange's equations, which power (as well as all the odd powers) was proven not to exist, and next enunciates the following general theorem: *If the elements of the first column of any odd-ordered circulant, axisymmetric with respect to the principal diagonal, be replaced by units, the sum of the complementary minors of the elements in the places (2, 2), (3, 3), ..., (n, n) vanishes.*

He next points out that in the case $n = 7$ we may substitute for

$$[2, 2]_1 + [3, 3]_1 + [4, 4]_1 + [5, 5]_1 + [6, 6]_1 + [7, 7]_1 = 0,$$

the two relations

$$[2, 2]_1 + [3, 3]_1 + [5, 5]_1 = 0,$$

$$[4, 4]_1 + [6, 6]_1 + [7, 7]_1 = 0,$$

where $[p, q]_r$ denotes the complementary minor of the element in the p th row and q th column after the r th column of the circulant has been replaced by units.

¹ For the purposes of this paper the following definition will be assumed: If each row of a determinant may be derived from the preceding row by passing the first element over all the others to the last place, the determinant is called a circulant.

² "Lagrange's determinantal equation in the case of a circulant," *Messenger of Mathematics*, New Series, vol. 41, March, 1912.

The object of this note is to investigate these aggregates a little more closely and from a somewhat different viewpoint, and to determine those that are fundamental.

By equating the coefficients of odd powers of x to zero we would get various vanishing aggregates connecting minors of even order but these aggregates would not be fundamental.

Starting with the determinant

$$\begin{vmatrix} b & c+a & d+e & 1 \\ c & d+b & e+a & 1 \\ d & e+c & a+b & 1 \\ e & a+d & b+c & 1 \end{vmatrix},$$

which obviously vanishes since two columns may be made identical, we have

$$\begin{vmatrix} b & c & d & 1 \\ c & d & e & 1 \\ d & e & a & 1 \\ e & a & b & 1 \end{vmatrix} + \begin{vmatrix} b & c & e & 1 \\ c & d & a & 1 \\ d & e & b & 1 \\ e & a & c & 1 \end{vmatrix} + \begin{vmatrix} b & a & d & 1 \\ c & b & e & 1 \\ d & c & a & 1 \\ e & d & b & 1 \end{vmatrix} + \begin{vmatrix} b & a & e & 1 \\ c & b & a & 1 \\ d & c & b & 1 \\ e & d & c & 1 \end{vmatrix} = 0,$$

or

$$[1, 4]_5 - [1, 5]_3 + [1, 2]_4 - [1, 3]_2 = 0. \quad (1)$$

From the properties of circulants of odd order we have

1. When $1+r$ is even

$$\begin{aligned} [1, r]_s &= \left[\frac{1+r}{2}, \frac{1+r}{2} \right]_{s-[(r-1)/2]} \quad \text{if } s > \frac{r-1}{2}, \\ &= \left[\frac{1+r}{2}, \frac{1+r}{2} \right]_{n+s-[(r-1)/2]} \quad \text{if } s \equiv \frac{r-1}{2}. \end{aligned}$$

2. When $1+r$ is odd

$$\begin{aligned} [1, r]_s &= - \left[\frac{n+r+1}{2}, \frac{n+r+1}{2} \right]_{s-[(n+r-1)/2]} \quad \text{if } s > \frac{n+r-1}{2}, \\ &= - \left[\frac{n+r+1}{2}, \frac{n+r+1}{2} \right]_{s+[(n-r+1)/2]} \quad \text{if } s \equiv \frac{n+r-1}{2}. \end{aligned}$$

We also have

1. When $r+s$ is even

$$\begin{aligned} [r, r]_s &= - \left[\frac{r+s}{2}, \frac{r+s}{2} \right]_{n+[(3r-s)/2]} \quad \text{if } 3r \equiv s, \\ &= - \left[\frac{r+s}{2}, \frac{r+s}{2} \right]_{(3r-s)/2} \quad \text{if } 3r > s; \end{aligned}$$

2. When $r + s$ is odd

$$\begin{aligned} [r, r]_s &= - \left[\frac{n+r+s}{2}, \frac{n+r+s}{2} \right]_{n+[(3r-n-s)/2]} \quad \text{if } 3r \equiv s + n, \\ &= - \left[\frac{n+r+s}{2}, \frac{n+r+s}{2} \right]_{(3r-n-s)/2} \quad \text{if } 3r > s + n. \end{aligned}$$

Making use of these relations (1) takes the form

$$[2, 2]_1 + [3, 3]_1 + [4, 4]_1 + [5, 5]_1 = 0,$$

which is as given by Muir.

If we had started with

$$\begin{vmatrix} b & c+e+a & d & 1 \\ c & d+a+b & e & 1 \\ d & e+b+c & a & 1 \\ e & a+c+d & b & 1 \end{vmatrix},$$

which is also equal to zero, we would have

$$[1, 4]_5 - [1, 5]_2 + [1, 2]_4 = 0. \quad (2)$$

Similarly

$$[1, 3]_5 + [1, 5]_2 - [1, 2]_3 = 0. \quad (3)$$

From (2) and (3) we get

$$[1, 3]_5 - [1, 2]_3 + [1, 4]_5 + [1, 2]_4 = 0,$$

or

$$[3, 3]_1 + [2, 2]_1 + [5, 5]_1 + [4, 4]_1 = 0,$$

which is (1).

Again

$$[1, 1]_2 + [3, 3]_2 + [4, 4]_2 + [5, 5]_2 = 0;$$

and since $[5, 5]_1 = -[3, 3]_2$, we have

$$[1, 1]_2 + [4, 4]_2 + [5, 5]_2 + [2, 2]_1 + [3, 3]_1 + [4, 4]_1 = 0. \quad (4)$$

The general law for (2) is, in the case of circulants of odd order,

$$(-1)^{r-1}[1, r]_s + (-1)^{s-1}[1, s]_t + (-1)^{t-1}[1, t]_r = 0. \quad (5)$$

The general law for (1) is

$$(-1)^{r-1}[1, r]_s + (-1)^{s-1}[1, s]_t + (-1)^{t-1}[1, t]_u + (-1)^{u-1}[1, u]_r = 0, \quad (6)$$

but, as has been seen, (6) is made up of two of (5).

The method here used will give vanishing aggregates of minors of any order. Thus from

$$\begin{vmatrix} 1 & a+b+c+d & e \\ 1 & b+c+d+e & a \\ 1 & c+d+e+a & b \end{vmatrix} = 0,$$

where as before, a, b, c, d, e are the elements of a circulant of order five, we have

$$\begin{vmatrix} 1 & a & e \\ 1 & b & a \\ 1 & c & b \end{vmatrix} + \begin{vmatrix} 1 & b & e \\ 1 & c & a \\ 1 & d & b \end{vmatrix} + \begin{vmatrix} 1 & c & e \\ 1 & d & a \\ 1 & e & b \end{vmatrix} + \begin{vmatrix} 1 & d & e \\ 1 & e & a \\ 1 & a & b \end{vmatrix} = 0,$$

or

$$- \begin{bmatrix} 45 \\ 23 \end{bmatrix}_4 + \begin{bmatrix} 45 \\ 34 \end{bmatrix}_1 - \begin{bmatrix} 45 \\ 41 \end{bmatrix}_2 + \begin{bmatrix} 45 \\ 12 \end{bmatrix}_3 = 0,$$

or

$$- \begin{bmatrix} 12 \\ 12 \end{bmatrix}_5 + \begin{bmatrix} 12 \\ 12 \end{bmatrix}_4 - \begin{bmatrix} 15 \\ 15 \end{bmatrix}_3 + \begin{bmatrix} 15 \\ 15 \end{bmatrix}_2 = 0,$$

where $\begin{bmatrix} r & s \\ t & u \end{bmatrix}_v$, is the complementary of the minor of the elements in the intersection of the r th and s th rows, and the t th and u th columns, after the elements in the v th column of the circulant are replaced by units.

Using this same method we may also obtain vanishing aggregates of minors of circulants of even order. Thus for $n = 4$ we have

$$\begin{vmatrix} 1 & a+b+c & d \\ 1 & b+c+d & a \\ 1 & c+d+a & b \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} 1 & a & d \\ 1 & b & a \\ 1 & c & b \end{vmatrix} + \begin{vmatrix} 1 & b & d \\ 1 & c & a \\ 1 & d & b \end{vmatrix} + \begin{vmatrix} 1 & c & d \\ 1 & d & a \\ 1 & a & b \end{vmatrix} = 0,$$

or

$$- [1, 1]_2 + [1, 2]_4 + [1, 4]_1 = 0,$$

which is in accord with (5).

Again

$$\begin{vmatrix} 1 & a+c & b+d \\ 1 & b+d & c+a \\ 1 & c+a & d+b \end{vmatrix} = 0,$$

and

$$[1, 2]_3 - [1, 3]_4 + [1, 4]_1 - [1, 1]_2 = 0,$$

which is in accord with (6). It is otherwise evident that we may remove from (5) and (6) the restriction that n must be odd, and have them true for any order of circulant.

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